

# STABLY ISOMORPHIC DUAL OPERATOR ALGEBRAS

G.K. ELEFThERAKIS AND V.I. PAULSEN

**ABSTRACT.** We prove that two unital dual operator algebras  $A, B$  are stably isomorphic if and only if they are  $\Delta$ -equivalent [7], if and only if they have completely isometric normal representations  $\alpha, \beta$  on Hilbert spaces  $H, K$  respectively and there exists a ternary ring of operators  $\mathcal{M} \subset B(H, K)$  such that  $\alpha(A) = [\mathcal{M}^* \beta(B) \mathcal{M}]^{-w^*}$  and  $\beta(B) = [\mathcal{M} \alpha(A) \mathcal{M}^*]^{-w^*}$ .

## 1. INTRODUCTION

Two dual operator algebras [1, 9]  $A, B$  are called **stably isomorphic** if there exists a cardinal  $I$  such that the algebras  $M_I(A), M_I(B)$  of matrices indexed by  $I$ , whose finite submatrices have uniformly bounded norms, are algebraically isomorphic through an isomorphism which is completely isometric and  $w^*$ -(bi)continuous. In the special case of  $W^*$ -algebras [1], this happens if and only if  $A$  and  $B$  are Morita equivalent in the sense of Rieffel [11]. A proof of this fact for separably acting von Neumann algebras can be found in [12] and the general case is in [1].

In [6, 7] two new equivalence relations between dual operator algebras were defined:

**Definition 1.1.** [6] *Let  $A, B$  be  $w^*$ -closed algebras acting on Hilbert spaces  $H$  and  $K$ , respectively. If there exists a **ternary ring of operators (TRO)**  $\mathcal{M} \subset B(H, K)$ , i.e. a subspace satisfying  $\mathcal{M}\mathcal{M}^*\mathcal{M} \subset \mathcal{M}$ , such that  $A = [\mathcal{M}^*B\mathcal{M}]^{-w^*}$  and  $B = [\mathcal{M}\mathcal{M}^*A]^{-w^*}$  we write  $A \overset{\mathcal{M}}{\approx} B$ . We say that the algebras  $A, B$  are **TRO equivalent** if there exists a TRO  $\mathcal{M}$  such that  $A \overset{\mathcal{M}}{\approx} B$ .*

If  $A$  is a dual operator algebra, then we call a completely contractive,  $w^*$ -continuous homomorphism  $\alpha : A \rightarrow B(H)$  where  $H$  is a Hilbert space, a **normal representation of  $A$** .

---

*Key words and phrases.* Morita equivalence, stable isomorphism, ternary ring.

This project is cofunded by European Social Fund and National Resources - (EPEAEK II) "Pythagoras II" grant No. 70/3/7997.

In [7] the notion of  $\Delta$ -equivalence of two unital dual operator algebras  $A, B$  was defined in terms of equivalence of two appropriate categories. In the present paper, we will adopt the following definition of  $\Delta$ -equivalence.

**Definition 1.2.** *Two unital dual operator algebras  $A, B$  are called  $\Delta$ -equivalent if they have completely isometric normal representations  $\alpha, \beta$  such that the algebras  $\alpha(A), \beta(B)$  are TRO equivalent.*

**Remark 1.1.** *The conclusion of the present paper (Theorem 3.2) was used in [7, Theorem 1.3]. It was proved in that theorem that definition 1.2 is in fact equivalent to the one given in [7, Definition 1.4]: there, two unital dual operator algebras  $A$  and  $B$  are called  $\Delta$ -equivalent if there exists an equivalence functor between their categories of normal representations which intertwines not only the representations of the algebras but also their restrictions to the diagonals.*

Two completely isometrically and  $w^*$ -continuously isomorphic unital dual operator algebras are not necessarily TRO equivalent, but they are  $\Delta$ -equivalent. Also two  $W^*$ -algebras are Morita equivalent in the sense of Rieffel if and only if they are  $\Delta$ -equivalent [7]. In this work we are going to prove that two unital dual operator algebras are  $\Delta$ -equivalent if and only if they are stably isomorphic.

We explain now why two stably isomorphic unital dual operator algebras are  $\Delta$ -equivalent. We need first to present some definitions and results, see for example [1]. If  $I$  is a cardinal and  $X$  is a dual operator space, we denote by  $\Omega_I(X)$  the linear space of all matrices with entries in  $X$ . If  $x \in \Omega_I(X)$  and  $r$  is a finite subset of  $I$  we write  $x^r = (x_{ij})_{i,j \in r}$ . We define

$$\|x\| = \sup_{r \subset I, \text{finite}} \|x^r\| \text{ and } M_I(X) = \{x \in \Omega_I(X), \|x\| < +\infty\}.$$

This space is a dual operator space. If  $X$  is a dual operator algebra then  $M_I(X)$  is also a dual operator algebra. In case  $X$  is a  $w^*$ -closed subspace of  $B(H, K)$  for some Hilbert spaces  $H, K$  we naturally identify  $M_I(X)$  as a subspace of  $B(H^I, K^I)$  where  $H^I$  (*resp.*  $K^I$ ) is the direct sum of  $I$  copies of  $H$  (*resp.*  $K$ ). We denote the  $w^*$ -closed subspace of  $B(H^I, K)$  consisting of bounded operators of the form

$$H^I \rightarrow K : (\xi_i)_{i \in I} \rightarrow \sum_i x_i(\xi_i)$$

for  $\{x_i : i \in I\} \subset X$  by  $R_I^w(X)$  and the  $w^*$ -closed subspace of  $B(H, K^I)$  consisting of bounded operators of the form

$$H \rightarrow K^I : \xi \rightarrow (x_i(\xi))_{i \in I}$$

for  $\{x_i : i \in I\} \subset X$  by  $C_I^w(X)$ . Observe that if  $X$  is a  $w^*$ -closed TRO then the spaces  $R_I^w(X), C_I^w(X)$  are  $w^*$ -closed TRO's.

Suppose now that the unital dual operator algebras  $A_0, B_0$  are stably isomorphic for a cardinal  $I$ . By [9] there exist completely isometric normal representations of  $A_0, B_0$  whose images we denote by  $A, B$ , respectively. Observe that the algebras  $A, M_I(A)$  are TRO equivalent, indeed,  $A \overset{\mathcal{M}}{\sim} M_I(A)$ , where  $\mathcal{M} = C_I^w(\Delta(A))$ , and  $\Delta(A) = A \cap A^*$  is the diagonal of  $A$ . Similarly the algebras  $B, M_I(B)$  are TRO equivalent. Since  $\Delta$ -equivalence is an equivalence relation preserved by normal completely isometric homomorphisms we conclude that the initial algebras are  $\Delta$ -equivalent.

The purpose of this paper is to prove the converse:  $\Delta$ -equivalent algebras are stably isomorphic. Since every completely isometric normal homomorphism  $A \rightarrow B$  for dual operator algebras naturally “extends” to a completely isometric normal homomorphism  $M_I(A) \rightarrow M_I(B)$  for every cardinal  $I$  [1], it suffices to show that the TRO equivalent algebras are stably isomorphic.

## 2. GENERATED BIMODULES.

In this section we prove that if  $A$  (*resp.*  $B$ ) is a  $w^*$ -closed subalgebra of  $B(H)$  (*resp.*  $B(K)$ ) for a Hilbert space  $H$  ( $K$ ) and  $\mathcal{M} \subset B(H, K)$  is a TRO such that  $A \overset{\mathcal{M}}{\sim} B$ , then there exist bimodules  $X, Y$  over these algebras, i.e.,  $AXB \subset X$ ,  $BYA \subset Y$ , which are generated by  $\mathcal{M}$ , such that  $A \cong X \overset{\sigma h}{\otimes}_B Y$  and  $B \cong Y \overset{\sigma h}{\otimes}_A X$  as dual spaces, where  $X \overset{\sigma h}{\otimes}_B Y$  ( $Y \overset{\sigma h}{\otimes}_A X$ ) is an appropriate quotient of the normal Haagerup tensor product  $X \overset{\sigma h}{\otimes} Y$  ( $Y \overset{\sigma h}{\otimes} X$ ) [5].

We start with some definitions and symbols. If  $\Omega$  is a Banach space we denote by  $\Omega^*$  its dual. If  $X, Y, Z$  are linear spaces,  $n \in \mathbb{N}$  and  $\sigma : X \rightarrow Y$  is a linear map we denote again by  $\sigma$  the map  $M_n(X) \rightarrow M_n(Y) : (x_{ij}) \rightarrow (\sigma(x_{ij}))$ . If  $\phi : X \times Y \rightarrow Z$  is a bilinear map and  $n, p \in \mathbb{N}$  we denote again by  $\phi$  the map  $M_{n,p}(X) \times M_{p,n}(Y) \rightarrow M_n(Z) : ((x_{ij}), (y_{ij})) \rightarrow (\sum_{k=1}^p \phi(x_{ik}, y_{kj}))_{ij}$ . If  $X, Y$  are operator spaces we denote by  $CB(X, Y)$  the space of completely bounded maps from  $X$  to  $Y$  with the completely bounded norm. If  $Z$  is another operator space, a bilinear map  $\phi : X \times Y \rightarrow Z$  is called completely bounded [10] if there exists  $c > 0$  such that  $\|\phi(x, y)\| \leq c\|x\|\|y\|$  for all  $x \in M_{n,p}(X), y \in M_{p,n}(Y), n, p \in \mathbb{N}$ . The least such  $c$  is the completely bounded norm of  $\phi$  and it is denoted by  $\|\phi\|_{cb}$ . We write

$$CB(X \times Y, Z) = \{\phi : X \times Y \rightarrow Z, \phi \text{ is completely bounded}\}.$$

This is an operator space under the identification

$$M_n(CB(X \times Y, Z)) = CB(X \times Y, M_n(Z))$$

for all  $n \in \mathbb{N}$ .

We denote the Haagerup tensor product of  $X, Y$  by  $X \overset{h}{\otimes} Y$ . The map  $CB(X \times Y, Z) \rightarrow CB(X \overset{h}{\otimes} Y, Z) : \omega \rightarrow \tilde{\omega}$  given by  $\tilde{\omega}(x \otimes y) = \omega(x, y)$  for all  $x \in X, y \in Y$  is a complete isometry. If  $X, Y$  are dual operator spaces we denote by  $CB^\sigma(X, Y)$  the space of completely bounded  $w^*$ -continuous maps. If  $Z$  is another dual operator space a bilinear map  $\phi : X \times Y \rightarrow Z$  is called **normal** if it is separately  $w^*$ -continuous. We denote by  $CB^\sigma(X \times Y, Z)$  the space of completely bounded normal bilinear maps.

We now recall the normal Haagerup tensor product [5]. In the rest of this section we fix dual operator spaces  $X, Y$  and the map

$$\pi : CB(X \times Y, \mathbb{C}) \rightarrow CB(X \overset{h}{\otimes} Y, \mathbb{C}) = (X \overset{h}{\otimes} Y)^*$$

given by  $\pi(\omega) = \tilde{\omega}, \tilde{\omega}(x \otimes y) = \omega(x, y)$ . We denote by  $\Omega_1$  the space  $\pi(CB^\sigma(X \times Y, \mathbb{C}))$  and by  $X \overset{\sigma h}{\otimes} Y$  the dual of  $\Omega_1$ . This space is the  $w^*$ -closed span of its elementary tensors  $x \otimes y, x \in X, y \in Y$  and it has the following property: For all dual operator spaces  $Z$  there exists a complete onto isometry

$$J : CB^\sigma(X \times Y, Z) \rightarrow CB^\sigma(X \overset{\sigma h}{\otimes} Y, Z) : \phi \rightarrow \phi_\sigma$$

where  $\phi_\sigma(x \otimes y) = \phi(x, y)$ .

We now fix a dual operator algebra  $B$  such that  $X$  is a right  $B$ -module and  $Y$  is left  $B$ -module and the maps

$$X \times B \rightarrow X : (x, b) \rightarrow xb, \quad B \times Y \rightarrow Y : (b, y) \rightarrow by$$

are complete contractions and normal bilinear maps. A bilinear map  $\omega : X \times Y \rightarrow Z$  is called  **$B$ -balanced** if  $\omega(xb, y) = \omega(x, by)$  for all  $x \in X, b \in B, y \in Y$ . For every dual operator space  $Z$  we define the space

$$CB^{B\sigma}(X \times Y, Z) = \{\omega \in CB^\sigma(X \times Y, Z) : \omega \text{ is } B\text{-balanced}\}.$$

We denote by  $\Omega_2$  the space  $\pi(CB^{B\sigma}(X \times Y, \mathbb{C}))$ . Observe that  $\Omega_2$  is a closed subspace of  $\Omega_1 \subset (X \overset{h}{\otimes} Y)^*$ . Also we define the space

$$N = [xb \otimes y - x \otimes by : x \in X, b \in B, y \in Y]^{-w^*} \subset X \overset{\sigma h}{\otimes} Y.$$

We denote by  $X \overset{\sigma h}{\otimes}_B Y$  the space  $(X \overset{\sigma h}{\otimes} Y)/N$  and we use the symbol  $x \otimes_B y$  for  $x \otimes y + N, x \in X, y \in Y$ .

**Proposition 2.1.** *The spaces  $X \overset{\sigma h}{\otimes}_B Y$  and  $\Omega_2^*$  are completely isometric and  $w^*$ -homeomorphic.*

**Proof.** The adjoint map  $\theta : X \overset{\sigma h}{\otimes}_B Y \rightarrow \Omega_2^*$  of the inclusion  $\Omega_2 \hookrightarrow \Omega_1$  is a complete quotient map and  $w^*$ -continuous. Check now that  $N = \text{Ker}(\theta)$ .  $\square$

**Proposition 2.2.** *If  $Z$  is a dual operator space and  $\phi \in CB^{B\sigma}(X \times Y, Z)$  then there exists a  $w^*$ -continuous and completely bounded map  $\phi_{B\sigma h} : X \overset{\sigma h}{\otimes}_B Y \rightarrow Z$  such that  $\phi_{B\sigma h}(x \otimes_B y) = \phi(x, y)$  for all  $x \in X, y \in Y$ . In fact the map  $CB^{B\sigma}(X \times Y, Z) \rightarrow CB^\sigma(X \overset{\sigma h}{\otimes}_B Y, Z) : \phi \rightarrow \phi_{B\sigma h}$  is a complete isometry, onto.*

**Proof.** Suppose that  $Z_*$  is the operator space predual of  $Z$ . For every  $\omega \in Z_*, \omega \circ \phi \in \Omega_2$ . So we can define a map  $\phi_* : Z_* \rightarrow \Omega_2 : \phi_*(\omega) = \omega \circ \phi$ . We denote by  $\phi_{B\sigma h}$  the adjoint map of  $\phi_*$ . So that  $\phi_{B\sigma h} \in CB(\Omega_2^*, Z) = CB(X \overset{\sigma h}{\otimes}_B Y, Z)$  by Proposition 2.1. For every  $x \in X, y \in Y, \omega \in Z_*$  we have  $\langle \phi_{B\sigma h}(x \otimes_B y), \omega \rangle = \langle \phi(x, y), \omega \rangle$  so  $\phi_{B\sigma h}(x \otimes_B y) = \phi(x, y)$ .

Let  $i : \Omega_2 \rightarrow \Omega_1$  denote the inclusion map so that  $q = i^* : \Omega_1^* \rightarrow \Omega_2^*$  is a  $w^*$ -continuous complete quotient map. The map of composition with  $q$  gives a completely isometric inclusion,  $q^* : CB^\sigma(\Omega_2^*, Z) \rightarrow CB^\sigma(\Omega_1^*, Z)$ .

By Proposition 2.1 we may identify  $\Omega_2^* = X \overset{\sigma h}{\otimes}_B Y$  and also we have  $\Omega_1^* = X \overset{\sigma h}{\otimes} Y$  by definition. Thus, modulo these identifications, we have that  $q^* : CB^\sigma(X \overset{\sigma h}{\otimes}_B Y, Z) \rightarrow CB^\sigma(X \overset{\sigma h}{\otimes} Y, Z)$  is a  $w^*$ -continuous complete isometry.

We also have that  $CB^{B\sigma}(X \times Y, Z) \subseteq CB^\sigma(X \times Y, Z)$  is a subspace endowed with the same matrix norms. Thus,  $J : CB^{B\sigma}(X \times Y, Z) \rightarrow CB^\sigma(X \overset{\sigma h}{\otimes}_B Y, Z)$  is also a completely isometric inclusion.

Now observe that  $J(\phi) = q^*(\phi_{B\sigma h})$ , so that  $\phi \rightarrow \phi_{B\sigma h}$  is a complete isometry and  $J(CB^{B\sigma}(X \times Y, Z)) \subseteq q^*(CB^\sigma(X \overset{\sigma h}{\otimes}_B Y, Z))$ .

It remains to show that the map is onto so that the above inclusion is an equality of sets. To see that  $\phi \rightarrow \phi_{B\sigma h}$  is onto  $CB^\sigma(X \overset{\sigma h}{\otimes}_B Y, Z)$ , let  $\tilde{\psi} \in CB^\sigma(X \overset{\sigma h}{\otimes}_B Y, Z)$  and  $\theta : X \overset{\sigma h}{\otimes} Y \rightarrow X \overset{\sigma h}{\otimes}_B Y : x \otimes y \rightarrow x \otimes_B y$  be the map in Proposition 2.1. Since  $\tilde{\psi} \circ \theta \in CB^\sigma(X \overset{\sigma h}{\otimes} Y, Z)$  the map  $\psi : X \times Y \rightarrow Z$  given by  $\psi(x, y) = \tilde{\psi} \circ \theta(x \otimes y) = \tilde{\psi}(x \otimes_B y)$  belongs to the space  $CB^\sigma(X \times Y, Z)$ . We have to prove that  $\psi$  is balanced.

If  $\omega \in Z_*$  then  $\omega \circ \tilde{\psi}$  belongs to the predual of  $X \overset{\sigma h}{\otimes}_B Y$ . So there exists  $\chi \in CB^{B\sigma}(X \times Y, \mathbb{C})$  such that  $\chi(x, y) = \omega(\psi(x, y))$  for all  $x \in X, y \in Y$ . Now for every  $x \in X, y \in Y, b \in B$  we have

$$\omega(\psi(xb, y)) = \chi(xb, y) = \chi(x, by) = \omega(\psi(x, by)).$$

The functional  $\omega$  is arbitrary in  $Z_*$  so  $\psi(xb, y) = \psi(x, by)$ . We have proved that the map  $CB^{B\sigma}(X \times Y, Z) \rightarrow CB^\sigma(X \overset{\sigma h}{\otimes}_B Y, Z) : \phi \rightarrow \phi_{B\sigma h}$  is an onto.  $\square$

Suppose now that  $H, K$  are Hilbert spaces,  $A$  and  $B$  are unital  $w^*$ -closed subalgebras of  $B(K)$  and  $B(H)$  respectively and  $\mathcal{M} \subset B(K, H)$  is a  $w^*$ -closed TRO such that  $A \overset{\mathcal{M}}{\sim} B$ .

**Definition 2.1.** *The spaces  $[A\mathcal{M}^*]^{-w^*}, [\mathcal{M}A]^{-w^*}$  are called the  $\mathcal{M}$ -generated  $A - B$  bimodules.*

In what follows we assume that  $X = [A\mathcal{M}^*]^{-w^*}, Y = [\mathcal{M}A]^{-w^*}$ . We can check that

$$X = [\mathcal{M}^*B]^{-w^*}, Y = [B\mathcal{M}]^{-w^*},$$

$$(2.1) \quad AXB \subset X, BYA \subset Y, A = [XY]^{-w^*}, B = [YX]^{-w^*}.$$

Let  $a \in A$ . We define a map

$$CB^{B\sigma}(X \times Y, \mathbb{C}) \rightarrow CB^{B\sigma}(X \times Y, \mathbb{C}) : \omega \rightarrow \omega_a,$$

by  $\omega_a(x, y) = \omega(x, ya)$ . This map is continuous. The adjoint map  $\pi_a : X \overset{\sigma h}{\otimes}_B Y \rightarrow X \overset{\sigma h}{\otimes}_B Y$  satisfies  $\pi_a(x \otimes_B y) = x \otimes_B (ya)$ . For every  $z \in X \overset{\sigma h}{\otimes}_B Y$  we define  $za = \pi_a(z)$ . Observe that if  $\left(\sum_{i=1}^{k_j} x_i^j \otimes_B y_i^j\right)_j$  is a net such that  $z = w^* - \lim_j \sum_{i=1}^{k_j} x_i^j \otimes_B y_i^j$  then  $za = w^* - \lim_j \sum_{i=1}^{k_j} x_i^j \otimes_B (y_i^j a)$ .

**Lemma 2.3.** *Let  $z \in X \overset{\sigma h}{\otimes}_B Y$ . If  $(a_\lambda)_\lambda \subset A$  is a net such that  $a_\lambda \xrightarrow{w^*} a$  then  $za_\lambda \xrightarrow{w^*} za$ .*

**Proof.** Choose  $\omega \in \text{Ball}(CB^\sigma(X \times Y, \mathbb{C}))$ . From the normal version of the Christensen, Sinclair, Paulsen, Smith theorem, see for example Theorem 5.1 in [5], there exist a Hilbert space  $H$  and normal completely contractive maps  $\phi_1 : X \rightarrow B(H, \mathbb{C}), \phi_2 : Y \rightarrow B(\mathbb{C}, H)$  such that  $\omega(x, y) = \phi_1(x)\phi_2(y)$ . Observe that the bilinear map  $Y \times A \rightarrow B(\mathbb{C}, H) : (y, a) \rightarrow \phi_2(ya)$  is completely contractive and normal. So by the same theorem there exist a Hilbert space  $K$  and complete contractions  $\phi_3 : A \rightarrow B(\mathbb{C}, K), \phi_4 : Y \rightarrow B(K, H)$

such that  $\phi_2(ya) = \phi_4(y)\phi_3(a)$  for all  $y \in Y, a \in A$ . The bilinear map  $X \times Y \rightarrow B(K, \mathbb{C}) : (x, y) \rightarrow \phi_1(x)\phi_4(y)$  is normal and a complete contraction. So there exists a completely contractive  $w^*$ -continuous map  $\pi : X \overset{\sigma h}{\otimes} Y \rightarrow B(K, \mathbb{C})$  such that  $\pi(x \otimes y) = \phi_1(x)\phi_4(y)$ . Now the map

$$\tau(\omega) : (X \overset{\sigma h}{\otimes} Y) \times A \rightarrow \mathbb{C} : \tau(\omega)(z, a) = \pi(z)\phi_3(a)$$

is normal, completely contractive and satisfies

$$\begin{aligned} \tau(\omega)(x \otimes y, a) &= \pi(x \otimes y)\phi_3(a) \\ &= \phi_1(x)\phi_4(y)\phi_3(a) = \phi_1(x)\phi_2(ya) = \omega(x, ya) \end{aligned}$$

for all  $x \in X, y \in Y, a \in A$ . The conclusion is that we can define a contraction

$$\tau : CB^\sigma(X \times Y, \mathbb{C}) \rightarrow CB^\sigma(X \overset{\sigma h}{\otimes} Y \times A, \mathbb{C}) : \omega \rightarrow \tau(\omega)$$

which has adjoint map  $\sigma : (X \overset{\sigma h}{\otimes} Y) \overset{\sigma h}{\otimes} A \rightarrow X \overset{\sigma h}{\otimes} Y$  satisfying  $\sigma((x \otimes y) \otimes a) = x \otimes (ya)$ . We recall from Proposition 2.1 the map

$$\theta : X \overset{\sigma h}{\otimes} Y \rightarrow X \overset{\sigma h}{\otimes}_B Y : \theta(x \otimes y) = x \otimes_B y.$$

Choose arbitrary  $z \in X \overset{\sigma h}{\otimes}_B Y$  and  $z_0 \in X \overset{\sigma h}{\otimes} Y$  such that  $\theta(z_0) = z$ . If  $\left(\sum_{i=1}^{k_j} x_i^j \otimes y_i^j\right)_j$  is a net such that  $z_0 = w^* - \lim \sum_{i=1}^{k_j} x_i^j \otimes y_i^j$  then for all  $a \in A$

$$\begin{aligned} \theta \circ \sigma(z_0 \otimes a) &= \theta \circ \sigma \left( \lim_j \left( \left( \sum_{i=1}^{k_j} x_i^j \otimes y_i^j \right) \otimes a \right) \right) = \lim_j \sum_{i=1}^{k_j} \theta(x_i^j \otimes (y_i^j a)) \\ &= \lim_j \sum_{i=1}^{k_j} x_i^j \otimes_B (y_i^j a) = za. \end{aligned}$$

If  $(a_\lambda)_\lambda \subset A$  is a net such that  $a_\lambda \xrightarrow{w^*} a$  then  $z_0 \otimes a_\lambda \xrightarrow{w^*} z_0 \otimes a$  in  $(X \overset{\sigma h}{\otimes} Y) \overset{\sigma h}{\otimes} A$ . Since  $\theta \circ \sigma$  is  $w^*$ -continuous we have  $\theta \circ \sigma(z_0 \otimes a_\lambda) \xrightarrow{w^*} \theta \circ \sigma(z_0 \otimes a)$  or equivalently  $za_\lambda \xrightarrow{w^*} za$ .  $\square$

**Theorem 2.4.**  $A \cong X \overset{\sigma h}{\otimes}_B Y$  and  $B \cong Y \overset{\sigma h}{\otimes}_A X$  completely isometrically and  $w^*$ -homeomorphically.

**Proof.** The map  $X \times Y \rightarrow A : (x, y) \rightarrow xy$  is normal, completely contractive and  $B$ -balanced. So by Proposition 2.2 it defines a completely contractive and  $w^*$ -continuous map

$$\pi : X \overset{\sigma h}{\otimes}_B Y \rightarrow A : \pi(x \otimes_B y) = xy.$$

We shall show that  $\pi$  is a complete isometry. Since  $A = [XY]^{-w*}$ , it will follow from the Krein Smulian theorem that  $\pi$  is onto  $A$ .

Let  $z = (z_{ij}) \in M_n(X \otimes_B^{\sigma h} Y)$ . It suffices to show that  $\|z\| \leq \|\pi(z)\|$ . Since  $X \otimes_B^{\sigma h} Y = (CB^{B\sigma}(X \times Y, \mathbb{C}))^*$  given  $\epsilon > 0$  there exist  $m \in \mathbb{N}$  and  $(\omega_{kl}) \in \text{Ball}(M_m(CB^{B\sigma}(X \times Y, \mathbb{C})))$  such that

$$\|z\| - \epsilon < \|((\omega_{kl}(z_{ij}))_{ij})_{kl}\|.$$

By Lemma 8.5.23 in [1] there exist partial isometries  $\{v_i : i \in I\} \subset \mathcal{M}$  with mutually orthogonal initial spaces such that  $I_H = \sum_{i \in I} \oplus v_i^* v_i$ . By the above lemma

$$w^* - \lim_{\substack{F \subset I \\ \text{finite}}} \sum_{s \in F} z_{ij} v_s^* v_s = z_{ij}$$

so

$$\lim_{\substack{F \subset I \\ \text{finite}}} \sum_{s \in F} \omega_{kl}(z_{ij} v_s^* v_s) = \omega_{kl}(z_{ij})$$

for all  $k, l, i, j$ . It follows that there exist partial isometries  $\{v_1, \dots, v_r\} \subset \mathcal{M}$  such that

$$\|z\| - \epsilon \leq \|(\sum_{s=1}^r \omega_{kl}(z_{ij} v_s^* v_s))_{ij}\|_{kl}.$$

Since  $X \otimes_B^{\sigma h} Y$  is the  $w^*$ -closure of the space  $(X \otimes Y)/N$ , see Proposition 2.1, there exists a net  $(z_\lambda)_\lambda \subset M_n(X \otimes Y/N)$  such that  $z_\lambda \xrightarrow{w^*} z$ . If  $z_\lambda = (z_{ij}(\lambda))_{ij}$  for all  $\lambda$  we have  $z_{ij}(\lambda) \xrightarrow{w^*} z_{ij}$ , hence  $\sum_{s=1}^r \omega_{kl}(z_{ij}(\lambda) v_s^* v_s) \rightarrow \sum_{s=1}^r \omega_{kl}(z_{ij} v_s^* v_s)$  for all  $i, j, k, l$ . It follows that there exists  $\lambda_0$  such that

$$\|z\| - \epsilon \leq \left\| \left( \left( \sum_{s=1}^r \omega_{kl}(z_{ij}(\lambda) v_s^* v_s) \right)_{ij} \right)_{kl} \right\| \quad \text{for all } \lambda \geq \lambda_0.$$

Fix  $i, j, \lambda$  and suppose that  $z_{ij}(\lambda) = \sum_{p=1}^t x_p \otimes_B y_p$ , then  $\omega_{kl}(z_{ij}(\lambda) v_s^* v_s) = \sum_{p=1}^t \omega_{kl}(x_p, y_p v_s^* v_s)$  for all  $k, l, s$ . Since  $y_p v_s^* \in YX \subset B$  and  $\omega_{kl}$  is  $B$ -balanced we have

$$\omega_{kl}(z_{ij}(\lambda) v_s^* v_s) = \sum_{p=1}^t \omega_{kl}(x_p y_p v_s^*, v_s) = \omega_{kl}(\pi(z_{ij}(\lambda)) v_s^*, v_s).$$

So we take the inequality

$$\|z\| - \epsilon \leq \left\| \left( \left( \sum_{s=1}^r \omega_{kl}(\pi(z_{ij}(\lambda)) v_s^*, v_s) \right)_{ij} \right)_{kl} \right\| \quad \text{for all } \lambda \geq \lambda_0.$$



Since  $\pi(z_{ij}(\lambda)) \xrightarrow{w^*} \pi(z_{ij})$  we have

$$\|z\| - \epsilon \leq \left\| \left( \left( \sum_{s=1}^r \omega_{kl}(\pi(z_{ij})v_s^*, v_s) \right)_{ij} \right)_{kl} \right\|_{mn}.$$

Let  $v = (v_1, \dots, v_r)^t$  and

$$x = (\pi(z_{ij}))_{ij} \cdot \begin{bmatrix} v^* & & \\ 0 & \ddots & 0 \\ & & v^* \end{bmatrix} \in M_{n,nr}(X), \quad y = \begin{bmatrix} v & & \\ 0 & \ddots & 0 \\ & & v \end{bmatrix} \in M_{nr,n}(Y).$$

The above inequality can be written in the following form

$$\|z\| - \epsilon \leq \|(\omega_{kl}(x, y))_{k,l}\|_{mn}.$$

Since

$$\|(\omega_{kl})\|_m = \|(\omega_{kl}) : X \times Y \rightarrow M_m\|_{cb} \leq 1$$

we have

$$\|z\| - \epsilon \leq \|x\| \|y\| \leq \|(\pi(z_{ij}))_{ij}\| \|v^*\| \|v\| \leq \|\pi(z)\|.$$

Since  $\epsilon > 0$  is arbitrary we obtain  $\|z\| \leq \|\pi(z)\|$ . This completes the proof of  $A \cong X \overset{\sigma h}{\otimes}_B Y$ . Similarly we can prove  $B \cong Y \overset{\sigma h}{\otimes}_A X$   $\square$

### 3. THE MAIN THEOREM

In this section we shall prove that two unital dual operator algebras are  $\Delta$ -equivalent if and only if they are stably isomorphic. As we noted in section 1 it suffices to show that TRO equivalent algebras are stably isomorphic. Thus in what follows, we fix unital  $w^*$ -closed algebras  $A, B$  acting on Hilbert spaces  $H, K$  respectively and a  $w^*$ -closed TRO  $\mathcal{M}$  such that  $A \overset{\mathcal{M}}{\sim} B$ . Let  $X = [A\mathcal{M}^*]^{-w^*}, Y = [\mathcal{M}A]^{-w^*}$  be the  $\mathcal{M}$ -generated  $A - B$  bimodules which satisfy (2.1). We give the following definition (see the analogous definition in [2]). If  $U_i \subset B(L, H), V_i \subset B(H, L), i = 1, 2$  are spaces such that  $U_i V_i \subset A, i = 1, 2$  a pair of maps  $\sigma : U_1 \rightarrow U_2, \pi : V_1 \rightarrow V_2$  is called **A-inner product preserving** if  $\sigma(x)\pi(y) = xy$  for all  $x \in U_1, y \in V_1$ .

**Lemma 3.1.** *There exist a cardinal  $I$  and completely isometric,  $w^*$ -continuous, onto,  $A$ -module maps  $\sigma : R_I^w(X) \rightarrow R_I^w(A), \pi : C_I^w(Y) \rightarrow C_I^w(A)$  such that the pair  $(\sigma, \pi)$  is  $A$ -inner product preserving.*

**Proof.** From Lemma 8.5.23 in [1] there exist partial isometries  $\{m_i : i \in I\} \subset \mathcal{M}$  with mutually orthogonal initial spaces and  $\{n_j : j \in J\} \subset \mathcal{M}$  with mutually orthogonal final spaces such that  $\sum_{i \in I} \oplus m_i^* m_i = I_H, \sum_{j \in J} \oplus n_j^* n_j = I_K$ .

By introducing sufficiently many 0 partial isometries to each set, we may assume that  $I^2 = I = J$ . We denote by  $m$  the column  $(m_i)_{i \in I} \in C_I^w(\mathcal{M})$ . We have  $m^*m = I_H$  and we denote by  $p$  the projection  $mm^* \in M_I(B)$ .

In what follows if  $U_n \subset B(H_n, K)$  are  $w^*$ -closed subspaces,  $H_n, K$  Hilbert spaces,  $n \in \mathbb{N}$ , we denote by  $U_1 \oplus_r U_2 \oplus_r \dots$  the  $w^*$ -closed subspace of  $B(\sum_n \oplus H_n, K)$  generated by the bounded operators of the form  $(u_1, u_2, \dots), u_n \in U_n, n \in \mathbb{N}$ . Also if  $V_n \subset B(K, H_n)$  are  $w^*$ -closed subspaces,  $H_n, K$  Hilbert spaces,  $n \in \mathbb{N}$  we denote by  $V_1 \oplus_c V_2 \oplus_c \dots$  the  $w^*$ -closed subspace of  $B(K, \sum_n \oplus H_n)$  generated by the bounded operators of the form  $(v_1, v_2, \dots)^t, v_n \in V_n, n \in \mathbb{N}$ . If  $(x_i)_{i \in I} \in R_I^w(R_I^w(X))$  where  $x_i \in R_I^w(X)$  then  $x_i m \in A$  and so we can define the maps

$$\begin{aligned} \tau_1 : R_I^w(R_I^w(X)) &\rightarrow R_I^w(A) \oplus_r R_I^w(R_I^w(X)p^\perp), \\ \tau_1((x_i)_{i \in I}) &= ((x_i m)_{i \in I}, (x_i p^\perp)_{i \in I}), \quad x_i \in R_I^w(X) \end{aligned}$$

and

$$\begin{aligned} \tau_2 : C_I^w(C_I^w(Y)) &\rightarrow C_I^w(A) \oplus_c C_I^w(p^\perp C_I^w(Y)), \\ \tau_2((y_i)_{i \in I}) &= ((m^* y_i)_{i \in I}, (p^\perp y_i)_{i \in I})^t, \quad y_i \in C_I^w(Y). \end{aligned}$$

This pair of maps is  $A$ -inner product preserving: if  $x \in R_I^w(R_I^w(X)), y \in C_I^w(C_I^w(Y))$  then

$$\tau_1(x)\tau_2(y) = (xm, xp^\perp)(m^*y, p^\perp y)^t = xmm^*y + xp^\perp y = xpy + xp^\perp y = xy.$$

These maps are onto because every  $a \in A$  may be written  $a = (am^*)m$  with  $am^* \in R_I^w(X)$  and also  $a = m^*(ma)$  with  $ma \in C_I^w(Y)$  and they are clearly  $w^*$ -continuous  $A$ -module maps. Also they are complete isometries. We check this fact for  $\tau_1$  and  $n = 2$ : If  $x = (x_{ij}) \in M_2(R_I^w(R_I^w(X)))$  we have

$$\begin{aligned} \|\tau_1(x)\|^2 &= \left\| \begin{bmatrix} x_{11}m & x_{11}p^\perp & x_{12}m & x_{12}p^\perp \\ x_{21}m & x_{21}p^\perp & x_{22}m & x_{22}p^\perp \end{bmatrix} \right\|^2 \\ &= \left\| \begin{bmatrix} x_{11}m & x_{12}m & x_{11}p^\perp & x_{12}p^\perp \\ x_{21}m & x_{22}m & x_{21}p^\perp & x_{22}p^\perp \end{bmatrix} \right\|^2 = \left\| x \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, x \begin{bmatrix} p^\perp & 0 \\ 0 & p^\perp \end{bmatrix} \right\|^2 \\ &= \left\| x \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} m^* & 0 \\ 0 & m^* \end{bmatrix} x^* + x \begin{bmatrix} p^\perp & 0 \\ 0 & p^\perp \end{bmatrix} x^* \right\|^2 = \|xx^*\| = \|x\|^2. \end{aligned}$$

We use the symbol  $\infty$  for the  $\aleph_0$  cardinal. The following spaces are isomorphic as  $A$ -modules and as dual operator spaces:

$$\begin{aligned} R_\infty^w(R_I^w(R_I^w(X))) &\cong R_I^w(A) \oplus_r R_I^w(R_I^w(X)p^\perp) \oplus_r R_I^w(A) \oplus_r \dots \\ &\cong R_I^w(A) \oplus_r R_\infty^w(R_I^w(R_I^w(X))) \end{aligned}$$

and

$$\begin{aligned} C_\infty^w(C_I^w(C_I^w(Y))) &\cong C_I^w(A) \oplus_c C_I^w(p^\perp C_I^w(YX)) \oplus_c C_I^w(A) \oplus_c \dots \\ &\cong C_I^w(A) \oplus_c C_\infty^w(C_I^w(C_I^w(Y))) \end{aligned}$$

Since  $I^2 = I$  it follows that  $\infty I = I$  so we have

$$R_I^w(X) \cong R_\infty^w(R_I^w(X)) \text{ and } C_I^w(Y) \cong C_\infty^w(C_I^w(Y)).$$

We conclude that there exist completely isometric,  $w^*$ -continuous,  $A$ -module bijections

$$\lambda_1 : R_I^w(X) \rightarrow R_I^w(A) \oplus_r R_I^w(X) \text{ and } \lambda_2 : C_I^w(Y) \rightarrow C_I^w(A) \oplus_c C_I^w(Y).$$

We can choose  $\lambda_1, \lambda_2$  to be  $A$ -inner product preserving. Similarly working with the partial isometries  $\{n_j : j \in I\}$  (see the beginning of the proof) we obtain a pair  $(\nu_1, \nu_2)$  of  $A$ -inner product preserving, completely isometric,  $w^*$ -continuous  $A$ -module bijections:

$$\nu_1 : R_I^w(A) \oplus_r R_I^w(X) \rightarrow R_I^w(A) \text{ and } \nu_2 : C_I^w(A) \oplus_c C_I^w(Y) \rightarrow C_I^w(A).$$

The maps

$$\sigma = \nu_1 \circ \lambda_1 : R_I^w(X) \rightarrow R_I^w(A) \text{ and } \pi = \nu_2 \circ \lambda_2 : C_I^w(Y) \rightarrow C_I^w(A)$$

satisfy our requirements.  $\square$

**Theorem 3.2.** *Two unital dual operator algebras are  $\Delta$  – equivalent if and only if they are stably isomorphic.*

**Proof.** It suffices to show that if the algebras,  $A$  and  $B$ , are TRO-equivalent, then they are stably isomorphic. Let  $I, \sigma, \pi$  be as in Lemma 3.1. Observe that  $A \overset{C_I^w(\mathcal{M})}{\sim} M_I(B)$  and the  $C_I^w(\mathcal{M})$ -generated  $A - M_I(B)$  bimodules (see definition 2.1) are the spaces  $R_I^w(X)$  and  $C_I^w(Y)$ . So by Theorem 2.4 the map

$$\psi_1 : C_I^w(Y) \overset{\sigma h}{\otimes}_A R_I^w(X) \rightarrow M_I(B) : \psi_1(y \otimes_A x) = yx$$

is a completely isometric,  $w^*$ -continuous bijection. For the same reason the map

$$\psi_2 : C_I^w(A) \overset{\sigma h}{\otimes}_A R_I^w(A) \rightarrow M_I(A) : \psi_2(a \otimes_A c) = ac$$

is a completely isometric,  $w^*$ -continuous bijection. The map

$$C_I^w(Y) \times R_I^w(X) \rightarrow C_I^w(A) \overset{\sigma h}{\otimes}_A R_I^w(A) : (y, x) \rightarrow \pi(y) \otimes_A \sigma(x)$$

is completely contractive, separately  $w^*$ -continuous and  $A$ -balanced. So by Proposition 2.2 there exists a completely contractive  $w^*$ -continuous map

$$C_I^w(Y) \overset{\sigma h}{\otimes}_A R_I^w(X) \rightarrow C_I^w(A) \overset{\sigma h}{\otimes}_A R_I^w(A) : y \otimes_A x \rightarrow \pi(y) \otimes_A \sigma(x).$$

We denote this map by  $\pi \otimes \sigma$ . Similarly we can define a complete contraction  $\pi^{-1} \otimes \sigma^{-1} : C_I^w(A) \overset{\sigma^h}{\otimes}_A R_I^w(A) \rightarrow C_I^w(Y) \overset{\sigma^h}{\otimes}_A R_I^w(X)$ . Since  $\pi^{-1} \otimes \sigma^{-1}$  is the inverse of  $\pi \otimes \sigma$  we conclude that  $\pi \otimes \sigma$  is a complete isometry. It follows that the map

$$\gamma = \psi_2 \circ (\pi \otimes \sigma) \circ \psi_1^{-1} : M_I(B) \rightarrow M_I(A)$$

is a completely isometric,  $w^*$ -continuous bijection. It remains to check that it is an algebraic homomorphism. Since  $M_I(B) = [C_I^w(Y)R_I^w(X)]^{-w^*}$  it suffices to show that  $\gamma(y_1x_1 \cdot y_2x_2) = \gamma(y_1x_1) \cdot \gamma(y_2x_2)$  for all  $x_1, x_2 \in R_I^w(X)$ ,  $y_1, y_2 \in C_I^w(Y)$ . Indeed,

$$\begin{aligned} & \gamma(y_1x_1 \cdot y_2x_2) \\ &= \psi_2 \circ (\pi \otimes \sigma) \circ \psi_1^{-1}(y_1x_1y_2 \cdot x_2) = (y_1x_1y_2 \in C_I^w(Y), x \in R_I^w(X)) \\ &= \psi_2 \circ (\pi \otimes \sigma)(y_1x_1y_2 \otimes_A x_2) = \psi_2(\pi(y_1x_1y_2) \otimes_A \sigma(x_2)) \\ &= \pi(y_1x_1y_2)\sigma(x_2) = (x_1y_2 \in A \text{ and } \pi \text{ is a } A\text{-module map}) \\ &= \pi(y_1)x_1y_2\sigma(x_2) = ((\sigma, \pi) \text{ is } A\text{-inner product preserving}) \\ &= \pi(y_1)\sigma(x_1)\pi(y_2)\sigma(x_2) = \psi_2(\pi(y_1) \otimes_A \sigma(x_1)) \cdot \psi_2(\pi(y_2) \otimes_A \sigma(x_2)) \\ &= \psi_2 \circ (\pi \otimes \sigma)(y_1 \otimes_A x_1) \cdot \psi_2 \circ (\pi \otimes \sigma)(y_2 \otimes_A x_2) \\ &= \psi_2 \circ (\pi \otimes \sigma) \circ \psi_1^{-1}(y_1x_1) \cdot \psi_2 \circ (\pi \otimes \sigma) \circ \psi_1^{-1}(y_2x_2) = \gamma(y_1x_1) \cdot \gamma(y_2x_2) \end{aligned}$$

□

**Remark 3.3.** *When the unital dual operator algebras  $A, B$  have completely isometric normal representations  $\alpha, \beta$  on separable, Hilbert spaces such that  $\alpha(A)$  and  $\beta(B)$  are TRO equivalent, then the proof of the above theorem shows that  $M_\infty(A)$  and  $M_\infty(B)$  are completely isometrically isomorphic, i.e., the index set  $I$  may be taken to be countable.*

#### 4. STABLY ISOMORPHIC CSL ALGEBRAS.

In this section we assume that all Hilbert spaces are separable. A set of projections on a Hilbert space is called a **lattice** if it contains the zero and identity operators and is closed under arbitrary suprema and infima. If  $A$  is a subalgebra of  $B(H)$  for some Hilbert space  $H$ , the set

$$\text{Lat}(A) = \{l \in pr(B(H)) : l^\perp Al = 0\}$$

is a lattice. Dually if  $\mathcal{L}$  is a lattice the space

$$\text{Alg}(\mathcal{L}) = \{a \in B(H) : l^\perp al = 0 \ \forall \ l \in \mathcal{L}\}$$

is an algebra. A commutative subspace lattice (**CSL**) is a projection lattice  $\mathcal{L}$  whose elements commute; the algebra  $\text{Alg}(\mathcal{L})$  is called a **CSL algebra**.

Let  $\mathcal{L}$  be a CSL and  $l \in \mathcal{L}$ . We denote by  $l_\flat$  the projection  $\vee\{r \in \mathcal{L} : r < l\}$ . Whenever  $l_\flat < l$  we call the projection  $l - l_\flat$  an **atom** of  $\mathcal{L}$ . If the CSL  $\mathcal{L}$  has no atoms we say that it is a **continuous CSL**. If the atoms span the identity operator we say that  $\mathcal{L}$  is a **totally atomic CSL**.

If  $\mathcal{L}_1, \mathcal{L}_2$  are CSL's,  $\phi : \mathcal{L}_1 \rightarrow \mathcal{L}_2$  is a **lattice isomorphism** (a bijection which preserves order) and  $p$  (resp.  $q$ ) is the span of the atoms of  $\mathcal{L}_1$  (resp. of  $\mathcal{L}_2$ ) there exists a well defined lattice isomorphism  $\mathcal{L}_1|_p \rightarrow \mathcal{L}_2|_q : l|_p \rightarrow \phi(l)|_q$  (Lemma 5.3 in [6].) Observe that the CSL's  $\mathcal{L}_1|_{p^\perp}, \mathcal{L}_2|_{q^\perp}$  are continuous. But it is not always true that  $\phi$  induces a lattice isomorphism from  $\mathcal{L}_1|_{p^\perp}$  onto  $\mathcal{L}_2|_{q^\perp}$ . In [3, 7.19] there exists an example of isomorphic nests  $\mathcal{L}_1, \mathcal{L}_2$  such that  $p^\perp = 0$  and  $q^\perp \neq 0$ . This motivates the following definition:

**Definition 4.1.** [6] *Let  $\mathcal{L}_1, \mathcal{L}_2$  be CSL's,  $\phi : \mathcal{L}_1 \rightarrow \mathcal{L}_2$  be a lattice isomorphism,  $p$  the span of the atoms of  $\mathcal{L}_1$  and  $q$  the span of the atoms of  $\mathcal{L}_2$ . We say that  $\phi$  **respects continuity** if there exists a lattice isomorphism  $\mathcal{L}_1|_{p^\perp} \rightarrow \mathcal{L}_2|_{q^\perp}$  such that  $l|_{p^\perp} \rightarrow \phi(l)|_{q^\perp}$  for every  $l \in \mathcal{L}_1$ .*

The following was proved in [6] (Theorem 5.7).

**Theorem 4.1.** *Let  $\mathcal{L}_1, \mathcal{L}_2$  be separably acting CSL's. The algebras  $\text{Alg}(\mathcal{L}_1), \text{Alg}(\mathcal{L}_2)$  are TRO equivalent if and only if there exists a lattice isomorphism  $\phi : \mathcal{L}_1 \rightarrow \mathcal{L}_2$  which respects continuity.*

Also we recall Theorem 3.2 in [8].

**Theorem 4.2.** *Two CSL algebras are  $\Delta$ -equivalent if and only if they are TRO equivalent.*

Combining Theorems 4.1, 4.2 with Theorem 3.2 we obtain the following:

**Theorem 4.3.** *Two CSL algebras, acting on separable Hilbert spaces, are stably isomorphic if and only if there exists a lattice isomorphism between their lattices which respects continuity.*

**Remark 4.4.** *In fact, since the CSL algebras, say  $\text{Alg}(\mathcal{L}_i), i = 1, 2$  are acting on separable Hilbert spaces, we have that if there exists a lattice isomorphism between  $\mathcal{L}_1$  and  $\mathcal{L}_2$  that respects continuity, then  $M_\infty(\text{Alg}(\mathcal{L}_1))$  and  $M_\infty(\text{Alg}(\mathcal{L}_2))$  are completely isometrically isomorphic.*

A consequence of this theorem is that two separably acting CSL algebras with continuous or totally atomic lattices are stably isomorphic if and only if they have isomorphic lattices.

## REFERENCES

- [1] D.P. Blecher, C. Le Merdy, Operator algebras and their modules, *London Mathematical Society Monographs*, 2004.
- [2] D.P. Blecher, P.S. Muhly, V.I. Paulsen, Categories of operator modules-Morita equivalence and projective modules, *Memoirs of the A.M.S.* 143 (2000) No 681.
- [3] Kenneth R. Davidson, *Nest algebras*, volume 191 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1988. Triangular forms for operator algebras on Hilbert space.
- [4] E.G. Effros and Z-J Ruan, Operator spaces, London Mathematical Society Monographs, New series 23, The Clarendon Press, Oxford University Press, New York, 2000.
- [5] E.G. Effros and Z-J Ruan, Operator space tensor products and Hopf convolution algebras, *J. Operator Theory* 50(2003), 131-156.
- [6] G.K. Eleftherakis, TRO equivalent algebras, preprint, ArXiv:math.OA/0607488
- [7] G.K. Eleftherakis, A Morita type equivalence for dual operator algebras, *Journal of Pure and Applied Algebra* (to appear), ArXiv:math.OA/0607489v4
- [8] G.K. Eleftherakis, Morita type equivalences and reflexive algebras, Arxiv:math.OA/0709.0600.
- [9] Christian Le Merdy, An operator space characterization of dual operator algebras, *American Journal of Mathematics*, 121 (1999), 55-63.
- [10] V.I. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge Studies in Advanced Math. 78, Cambridge University Press, Cambridge, 2002.
- [11] M.A. Rieffel, Morita equivalence for  $C^*$ -algebras and  $W^*$ -algebras, *Journal of Pure and Applied Algebra* 5: 51-96, 1974.
- [12] Zhong-Jin Ruan, Type decomposition and the Rectangular AFD property for  $W^*$ -TRO's, *Canad. J. Math.*, 56(2004), no 4, 843-870.

DEPT. OF MATHEMATICS, UNIVERSITY OF ATHENS, ATHENS, GREECE  
*E-mail address:* gelefth@math.uoa.gr

DEPT. OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TX, 77204  
*E-mail address:* vern@math.uh.edu